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On the Propagation of Massless Fields

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Abstract

An inequality is discussed expressing the idea that a physical quantity associated with massless fields should 'propagate at the speed of light'. It is shown that the inequality holds for the energy of massless bosons of spin zero and one.

1. *Introduction*

The non-existence of a real, bilinear, covariant and conserved probability current density $j^{\mu}(x)$ for free massless bosons and energy density tensor $T^{\mu\nu}(x)$ for free massless fermions such that $j^0(x) \geqslant 0$ and $T^{00}(x) \geqslant 0$ (Rashid, 1970) has led Petzold and Gerlach to formulate an inequality which they call 'a criterion for the mass of a free field to be zero' (Petzold & Gerlach, 1971). Their reasoning is similar to that on which the formulation of the concept of 'macrocausality' was based in connection with the propagation of (positive) physical quantities related to fields with non-zero mass (Gerlach *et al.*, 1967a; Gerlach, 1967b).

In this paper we present a more general statement of Petzold's inequality and show that despite the rather classical argument used to formulate it, the inequality holds for certain well-known (positive) densities related to free massless fields.

2. *Formulation of the Inequality*

Consider an ensemble of free massless point particles moving at the speed of light along straight lines. All particles at the point $\mathbf{x}$ at time x^0 will at time $x^{0\prime}$ be on the sphere

$$S_{x^0,\mathbf{x}}^{x^{0\prime}} = \{\mathbf{x}' \mid (\mathbf{x} - \mathbf{x}')^2 = (x^0 - x^{0\prime})^2\}$$

(the speed of light is set equal to unity). Thus all particles in a region F_{x^0} at time x^0 will be in the region

$$G_{x^{0\prime}} = \bigcup_{\mathbf{x} \in F_{x^0}} S_{x^0,\mathbf{x}}^{x^{0\prime}}$$

at time $x^{0'}$. The suffix x^0 is attached to the symbol denoting a region for the sake of clarity. Let $N(x^0, F_{x^0})$ denote the number of particles in F_{x^0} at time x^0 or their energy. By noticing that at time $x^{0'}$ there may be particles in $G_{x^{0'}}$ which at time x^0 occupied regions other than F_{x^0}, one may write down the inequality

$$N(x^{0'}, G_{x^{0'}}) \geqslant N(x^0, F_{x^0}) \tag{2.1}$$

In a sense (2.1) expresses the idea that the particles or their energy 'propagate at the speed of light'.† We show that the inequality holds for the energy of free massless bosons of spin zero and unity as well as for the position probability of free neutrinos.‡ Huygens' principle does not hold for the densities involved, since these are bilinear in the fields. We are aware of the fact that (2.1) has little to do with 'causality' in the sense of quantum physics since at no stage of the argument any measurement of position or energy (density) was performed. This is precisely the reason why we think the result to be of interest at least from a mathematical point of view.

3. *The Massless Complex Scalar Field*

In order to prove the inequality (2.1) for the energy of the massless complex scalar field we proceed as follows. Consider an arbitrary set of four square integrable functions $F'^A_\mu(x')$ at time $x^{0'}$ (not necessarily a four vector) and define the set of functions $F_\mu{}^A(x)$ at time x^0 by

$$F_n{}^A(x) = f_{n0}(x) + f_{0n}(x) \tag{3.1}$$

$$F_0{}^A(x) = \sum_{u=0}^{3} f_{\mu\mu}(x) \tag{3.2}$$

where

$$f_{\mu\gamma}(x) = \int D_{|\mu}(x - x')\, F'^A_\gamma(x')\, d^3 x' \tag{3.3}$$

$D(x)$ is the usual 'D function'§ and Greek indices run from zero to three, Latin indices from one to three. $D_{|\mu}(x)$ denotes the derivative of $D(x)$ with respect to x^μ. The quantity

$$T(x) = \sum_\gamma |F_\gamma{}^A(x)|^2 \tag{3.4}$$

has the same form as the energy density although it is not an energy density in general. Now one may write

$$T(x) = \sum_{\mu\gamma} |f_{\mu\gamma}(x)|^2 + Z(x) + S(x) + S^*(x) \tag{3.5}$$

† One may easily convince oneself that a necessary condition for the validity of (2.1) is that the relevant densities be non-negative. This is not very interesting however since the non-negativeness of the quantities integrated over arbitrary volumes was assumed in the formulation.

‡ Petzold & Gerlach (1971) formulated (2.1) only for the special case where F_{x^0} is a sphere and proved a much weaker form of the inequality.

§ See, for instance, Schweber (1964) for the propagators of free massless fields. Compendium 4, page 912.

where

$$Z(x) = \sum_{m,n} \{f^*_{nn}(x) f_{mm}(x) - f^*_{mn}(x) f_{mn}(x)\} \tag{3.6}$$

and

$$S(x) = \sum_{m} \{f^*_{m0}(x) f_{0m}(x) + f^*_{00}(x) f_{mm}(x)\} \tag{3.7}$$

The integral of $T(x)$ may be evaluated in terms of $F'^A_\gamma(x')$ by using various properties of $D(x)$.† The result is

$$\int S(x)\, d^3x = 0, \qquad \int Z(x)\, d^3x \leqslant 0 \tag{3.8}$$

and

$$\int \sum_{\mu,\gamma} |f_{\mu\gamma}(x)|^2\, d^3x = \int \sum_{\mu} |F'^A_\mu(x')|^2\, d^3x' \tag{3.9}$$

Hence we have the inequality

$$\int \sum_{\mu} |F'^A_\mu(x')|^2\, d^3x' \geqslant \int \sum_{\mu} |F_\mu{}^A(x)|^2\, d^3x \tag{3.10}$$

where the integrals extend over all space. At this stage consider a region $F^{\oplus}_{x0}$ properly containing F_{x^0} and define the region $G^{\oplus}_{x0'}$ in the same manner as $G_{x^{0'}}$ was defined earlier by starting with F_{x^0}. Let $\phi(x)$ be a complex scalar solution of the Klein–Gordon equation for zero mass for which the total energy is finite. Put $F'^A_\mu(x') = \phi_{|\mu}(x')$ for $\mathbf{x}' \in G_{x^{0'}}$ and zero for $\mathbf{x}' \notin G^{\oplus}_{x0'}$. Then it follows from the properties of $D(x)$ and its derivatives that $F^A_\mu(x) = \phi_{|\mu}(x)$ for $\mathbf{x} \in F_{x^0}$. Thus the inequality (3.10) becomes

$$\int_{G_{x^{0'}}} T^{00}(x')\, d^3x' \geqslant \int_{F_{x^0}} T^{00}(x)\, d^3x + o(\eta) \tag{3.11}$$

where $T^{00}(x) = \sum_\mu |\phi(x)|^2$ is the energy density and η denotes the difference in volume between $G^{\oplus}_{x0'}$ and $G_{x^{0'}}$ and tends to zero as $F^{\oplus}_{x0}$ tends to F_{x^0}. This completes the proof of the inequality (2.1) for the energy of the massless complex scalar field.

4. *The Energy of the Free Electromagnetic Field*

The proof of (2.1) for the energy of the free electromagnetic field is similar to that of the scalar field. Consider arbitrary square integrable functions $\mathbf{E}'^A(x')$, $\mathbf{H}'^A(x')$ at time $x^{0'}$ where $\mathbf{a} = (a_1, a_2, a_3)$. Define the functions $\mathbf{E}^A(x)$, $\mathbf{H}^A(x)$ at time x^0 by

$$\mathbf{E}^A(x) = \frac{\partial}{\partial x^0}\mathbf{e}(x) + \nabla\Lambda\mathbf{h}(x) \tag{4.1}$$

$$\mathbf{H}^A(x) = \frac{\partial}{\partial x^0}\mathbf{h}(x) - \nabla\Lambda\mathbf{e}(x) \tag{4.2}$$

† See, for instance, Schweber (1964) for the propagators of free massless fields. Compendium 4, page 912.

where

$$\mathbf{e}(x) = \int D(x - x')\,\mathbf{E}'^{A}(x')\,d^3x' \tag{4.3}$$

$$\mathbf{h}(x) = \int D(x - x')\,\mathbf{H}'^{A}(x')\,d^3x' \tag{4.4}$$

The quantity

$$\begin{aligned} T(x) &= \mathbf{E}^{A^2}(x) + \mathbf{H}^{A^2}(x) \\ &= \sum_{\mu}\left\{\frac{\partial}{\partial x^{\mu}}\mathbf{e}(x)\right\}^2 + \sum_{\mu}\left\{\frac{\partial}{\partial x^{\mu}}\mathbf{h}(x)\right\}^2 \\ &\quad + Z_1(x) + Z_2(x) + Z_3(x) \end{aligned} \tag{4.5}$$

has the form of an energy density, although it need not be an energy density in general. Here

$$\begin{aligned} Z_1(x) &= -\sum_{n}\left\{\frac{\partial}{\partial x^{n}}\mathbf{e}(x)\right\}^2 + \{\nabla\Lambda\mathbf{e}(x)\}^2 \\ Z_2(x) &= -\sum_{n}\left\{\frac{\partial}{\partial x^{n}}\mathbf{h}(x)\right\}^2 + \{\nabla\Lambda h(x)\}^2 \\ Z_3(x) &= 2\sum_{nme}\mathscr{E}_{nml}\left\{\frac{\partial}{\partial x^0}e_n(x)\frac{\partial}{\partial x^m}h_l(x) + \frac{\partial}{\partial x^m}e_n(x)\frac{\partial}{\partial x^0}h_l(x)\right\} \end{aligned} \tag{4.6}$$

and $\mathscr{E}_{nml}$ is the Levi–Cevita tensor. By using the properties of $D(x)$ we find that on integration the first two terms (4.5) reduce to $\int \mathbf{E}'^{A^2}(x')\,d^3x'$ and $\int \mathbf{H}'^{A^2}(x')\,d^3x'$ respectively while $\int Z_1(x)\,d^3x \leqslant 0$, $\int Z_2(x)\,d^3x \leqslant 0$ and $\int Z_3(x)\,d^3x = 0$. Hence we have

$$\int \{\mathbf{E}'^{A^2}(x') + \mathbf{H}'^{A^2}(x')\}\,d^3x' \geqslant \int \{\mathbf{E}^{A^2}(x) + \mathbf{H}^{A^2}(x)\}\,d^3x \tag{4.7}$$

Let $\mathbf{E}(x')$, $\mathbf{H}(x')$ be a solution of Maxwell's equations in empty space for which the total energy is finite. Proceeding as before, we put $\mathbf{E}'^{A}(x') = \mathbf{E}(x')$ and $\mathbf{H}'^{A}(x') = \mathbf{H}(x')$ for $\mathbf{x}' \in G_{x^{0\prime}}$ and zero outside $G^{\ddagger}_{x^{0\prime}}$. Then it follows from the properties of $D(x)$ that $\mathbf{E}^{A}(x) = \mathbf{E}(x)$ and $\mathbf{H}^{A}(x) = \mathbf{H}(x)$ for $\mathbf{x} \in F_{x^0}$. Hence, as before,

$$\int_{G_{x^{0\prime}}} T^{00}(x')\,d^3x' \geqslant \int_{F_{x^0}} T^{00}(x)\,d^3x + O(\eta)$$

where $\eta \to 0$ as $F^{\ddagger}_{x^0} \to F_{x^0}$ and $T^{00}(x) = \mathbf{E}^2(x) + \mathbf{H}^2(x)$ is the energy density.

5. *The Position Probability for Neutrinos*

We do not write out the proof of (2.1) for the position probability of neutrinos which is similar to the earlier procedure. There is, however, one difference: instead of the general inequalities (3.10) and (4.7) one obtains an equality. The result then is

$$\int_{G_{x^0'}} j^0(x')\,d^3x' \geqslant \int_{F_{x^0}} j^0(x)\,d^3x \qquad \text{where } j^0(x) = \bar{\psi}(x)\gamma^0\psi(x) \geqslant 0$$

and

$$\frac{\partial}{\partial x^0}\psi(x) = -\gamma^0\gamma\,.\,\nabla\psi(x), \qquad \int j^0(x)\,d^3x < \infty$$

6. *Final Remarks*

(a) The validity of (2.1) for the quantities discussed may seem to suggest that in some classical sense these quantities propagate at the speed of light. The inequality, however, is much more general than the classical argument used to formulate it.

(b) The validity of (2.1) is not trivial. In fact it is not difficult to use fields obeying Huygens' Principle to construct positive 'densities' violating the inequality.

(c) Since Huygens' Principle does not hold for solutions of the massless Klein–Gordon equation in an even number of space dimensions the inequality will not be valid, e.g. in two-dimensional problems.

(d) In one space dimension the result is trivial.

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